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Multifractality in Elastic Percolation

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We present numerical results on the distribution of forces in the central-force percolation model at threshold in two dimensions. We conjecture a relation between the multifractal spectrum of scalar and vector percolation that we test for central-foce percolation. This relation is in excellent agreement with our numerical data.

KEY WORDS: Percolation; multifractality; elasticity; critical exponents.

1. INTRODUCTION

Scalar transport properties in the framework of percolation are now well understood. In particular, the multifractal spectrum of the distribution of currents has been extensively studied.[1-3] Vector transport properties far less known. Two classes have been considered in the past:

Class I (referred to as AE in the following.[4] Systems having angular elasticity such as bond bending (e.g., in the beam lattice model) or angular springs.

Class II (referred to as CF in the following).[5] Systems where the only microscopic interactions are central force.

In the first class, systems are critical at the connectivity (or scalar) percolation threshold. Most studies of this problem have dealt with the scaling of the elastic modulus E, either directly[4,6] or through a related property (density of states of vibrations,[7] viscoelastic behavior, etc.). E vanishes when the concentration of bonds p approaches the threshold p_c as $E \propto (p - p_c)^{\tau}$. Here τ is very different from the scalar transport exponent t

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(in two dimensions, $t = 1.30 \pm 0.01$, whereas $\tau = 3.96 \pm 0.04$[6]). However, it has been suggested[8] that τ could be related to t through the equality $\tau = t + 2\nu$, where ν is the correlation length exponent. Although only an inequality has been shown,[9] the equality seems fulfilled numerically. In this paper, we generalize this conjecture to the complete multifractal spectrum of the AE percolation model.

The second class, CF percolation, has been much more controversial. Different numerical results exist in the literature suggesting that this problem belongs to a new universality class,[10] distinct from that of percolation, or even to no universality class at all.[11] In previous studies,[12] we have obtained, for different critical exponents, a good agreement with the values obtained for AE systems. We present here the multifractal spectrum of CF percolation lattices, extracted from extensive numerical simulations. Assuming now the indistinguishability of CF and AE critical properties, we verify that our conjectured relation between scalar and vector multifractal spectra agrees with our data in the CF percolation model.

2. MULTIFRACTALITY AT THE PERCOLATION THRESHOLD: SCALAR CASE

At the percolation threshold, the current distribution in random resistor networks is *multifractal*.[1,2,13] This means that each moment of it scales with an independent exponent $x(n)$, i.e.,

$$\langle i^n \rangle = \sum_k |i_k|^n \propto L^{x(n)} \tag{1}$$

where i_k is the current flowing through bond k, for any given boundary condition, and L is the size of the network. This expression is valid in the thermodynamic limit (L tending to infinity).

The scaling properties of the moments imply certain scaling features of the current distribution itself. This connection is given by relating the so-called f versus α function, to be defined below, to the scaling exponents $x(n)$. Let $N(i, L)\,\varepsilon$ be the number of bonds in a lattice of size L that carry a current between i and $i(1+\varepsilon)$, where ε is a small, fixed number. If this current distribution histogram $N(i, L)$ is rewritten $L^{f(i,L)}$ by defining the function $f(i, L) = \log[N(i, L)]/\log L$, the existence of a hierarchy of exponents $x(n)$ implies that $f(i, L)$ plotted versus $\alpha(i, L) = \log i/\log L$ will be independent of L for large L.

The nth current moment gets its contribution from the current distribution histogram at the point where $df(\alpha)/d\alpha = -n$. If the corresponding scaled current is called $\alpha(n)$, the following relation exists between this

current and the scaling exponent $x(n)$: $dx(n)/dn = \alpha(n)$. The shape of the scaled histogram $f(\alpha)$ can be related to the moment scaling exponent through $f(\alpha(n)) = x(n) - n\alpha(n)$. One may interpret $f(\alpha(n))$ as the fractal dimension of the set of bonds that contribute to the nth moment.

The fact that numerical simulations of random resistor networks can only be done on moderate-size lattices imposes the averaging over an ensemble of realizations. Some controllable quantity must be kept constant for each realization of the ensemble. The choice of this quantity defines the ensemble. For example, one defines the *constant-current* ensemble when the total current flowing into each lattice is the same, the *constant-dissipation* ensemble when the total energy dissipated in the resistors is kept constant, and the *constant-voltage* ensemble when the voltage difference across each lattice is identical. We refer to those three ensembles with subscripts c, e, and v, respectively. Among the three quantities introduced above $f(n)$, $\alpha(n)$, and $x(n)$, only $f(n)$ is independent of the ensemble chosen. However, exchanging ensembles corresponds to a translation of the α axis in the f–α plane. The relation between the above-mentioned ensembles is obtained in expressing in Eq. (1) the normalization of a given ensemble assuming that the currents are obtained in a different ensemble.[3] Some elementary algebra (see Appendix) gives

$$x_v(n) = x_c(n) - n x_c(2) \tag{2}$$

$$\alpha_v(n) = \alpha_c(n) - x_c(2) \tag{2'}$$

and

$$x_e(n) = x_c(n) - (n/2)\, x_c(2) \tag{3}$$

$$\alpha_e(n) = \alpha_c(n) - (1/2)\, x_c(2) \tag{3'}$$

The "natural" ensemble for the random resistor network is the constant-current one. In particular, for $n = 0$, $x_c(0)$ is the fractal dimension of the backbone, and for $n = 2$, $x_c(2)$ is t/ν (t is the conductivity exponent, ν is the correlation length exponent). The exponents $x_c(n)$ are listed in Table I. As can be seen, these exponents converge toward the value $1/\nu = 3/4$ as n tends to infinity.[14] This value is nothing but the fractal dimension of the cutting bonds (or singly connected bonds).[15]

3. MULTIFRACTALITY AT THE PERCOLATION THRESHOLD: VECTOR CASE

Let us consider now AE systems at the usual connectivity percolation threshold. For simplicity, let us assume, for instance, that we are dealing with a beam lattice. Each bond is a beam clamped to its neighbors at the

Table I. Estimates of Scaling Exponents

Moment order n	$x_c(n)$	$y_f(n)$	$y_f(n)^*$	$y_{ee}(n)$	$y_{ee}(n)^*$	$y_d(n)$	$y_d(n)^*$
0	1.62	1.65	1.62	1.65	1.62	1.65	1.62
1	1.19	2.12	2.19	0.67	0.70	−0.77	−0.78
2	0.98	4.0	2.98	−0.03	0.00	−2.9	−2.98
3	0.86	—[b]	3.86	−0.57	−0.61	−4.6	−5.07
4	0.78	—[b]	4.78	−1.02	−1.17	−6.4	−7.13
5	0.76	—[b]	5.76	−1.43	−1.68	—[b]	−9.13

[a] Scaling exponents for: the random resistor network in the constant-current ensemble, $x_c(n)$; the CF percolation model in the constant-force ensemble, $y_f(n)$ (measured value); the CF percolation model in the constant-force ensemble, $y_f(n)^*$ [expected value, Eq. (8)]; the CF percolation model in the constant-energy ensemble, $y_{ee}(n)$ (measured value); the CF percolation model in the constant-energy ensemble, $y_{ee}(n)^*$ [expected value, Eq. (8′)]; the CF percolation model in the constant-displacement ensemble, $y_d(n)$ (measured value); the CF percolation model in the constant-displacement ensemble, $y_d(n)^*$ [expected value, Eq. (8″)].
[b] The data are too ill behaved to allow any reliable determination of the exponents $y(n)$.

nodes of the lattice. A beam is, in general, subjected to different stresses: axial and transverse forces, and torques applied at its ends. Therefore, the procedure described above to obtain the multifractal spectrum through rescaled histograms is not straightforward. We propose to construct histograms from a scalar quantity which is representative of the stresses each beam is submitted to: namely, the elastic energy. More precisely, in order to be able to compare easily with other cases where the natural quantity is either the current (scalar case) or the axial force (CF case), we may study the distribution of $\sqrt{e_k}$, where e_k is the elastic energy of bond k. Now, as usual, the second moment of the distribution $\sum_k (\sqrt{e_k})^2$ is nothing but the total elastic energy of the lattice. For the central force case, the distribution of $\sqrt{e_k}$ is strictly equivalent to the distribution of forces.

This multifractal formalism can readily be applied to elastic percolation. The nth moment is given by

$$m_{(n)} = \sum_k (\sqrt{e_k})^n \propto L^{y(n)} \tag{4}$$

where e_k is the energy stored in the kth bond. There are three ensembles of elastic networks corresponding to the three previous ensembles for the random resistor network: the constant-displacement, the constant-energy, and the constant-force ensembles. We will refer to these with the subscripts d, ee, and f, respectively. The scaling exponents corresponding to these ensembles are related by equations similar to Eqs. (2) and (3):

$$y_d(n) = y_f(n) - ny_f(2) \tag{5}$$

$$\alpha_d(n) = \alpha_f(n) - y_f(2) \tag{5′}$$

and

$$y_{ee}(n) = y_f(n) - (n/2)\, y_f(2) \tag{6}$$

$$\alpha_{ee}(n) = \alpha_f(n) - (1/2)\, y_f(2) \tag{6'}$$

In particular, for $n = 0$, $y_f(0)$ is the fractal dimension of the backbone, and for $n = 2$, $y_f(2)$ is τ/ν (τ is the critical exponent of the elastic modulus, ν is the correlation length exponent).

4. CONJECTURE ABOUT THE RELATION BETWEEN SCALAR AND VECTOR CASES

For AE systems, the force-carrying part of the lattice, i.e., the backbone, can be schematized by the nodes-links-blobs picture (see, e.g., ref. 14). Within this framework, it is easy to see that most of the elastic energy of the structure is due to torques that will be of the order of ξf), where f is the typical force acting on a macrobond of length ξ, and ξ is the correlation length (mesh size of the nodes-links-blobs lattice). Moreover, the propagation of torques on a topologically one-dimensional system is identical, for each component, to that of a current. The combination of these two facts leads naturally to the relation[8]

$$\tau = t + 2\nu \tag{7}$$

that relates the scaling of the second moment in AE systems and in the scalar case. This equality, obtained thanks to very crude approximations, has been conjectured to be valid for AE systems. This is supported by numerical results.[6] More precisely, only the inequality $\tau \leqslant t + 2\nu$ can be shown.[8] A straightforward extension of this conjecture to any moment of the distribution is given by the formula

$$y_f(n) = x_c(n) + n \tag{8}$$

where the previous equality (7) is recovered for $n = 2$ since $y_f(2) = \tau/\nu$ and $x_c(2) = t/\nu$.

Such a relation has also been used implicitly in the limit when n tends to infinity (i.e., for the maximum stressed bonds) for the study of the rupture of elastic percolating structures.[16] However, only the case $n = 2$ has ever been tested experimentally or numerically.

Let us now examine the consequences of this conjecture for the different ensembles introduced previously. Using Eqs. (2), (3) and (5), (6), we obtain

$$y_{ee}(n) = x_e(n) \tag{8'}$$

$$y_d(n) = x_v(n) - n \tag{8''}$$

Let us first recall that for large n, $x_c(n)$ tends to the fractal dimension of singly connected bonds, namely $1/\nu$. Therefore, for large n,

$$y_f(n) = 1/\nu + n + o(1) \tag{9}$$

$$y_{ee}(n) = 1/\nu - [x_c(2)/2]\, n + o(1) \tag{9'}$$

$$y_d(n) = 1/\nu - [x_c(2) + 1]\, n + o(1) \tag{9''}$$

Now, concerning α, we obtain

$$\alpha_f = \alpha_c + 1 \tag{10}$$

$$\alpha_{ee} = \alpha_e \tag{10'}$$

$$\alpha_d = \alpha_v - 1 \tag{10''}$$

and, whatever the ensemble considered, the value of $f(n)$ for elastic problems is equal to the corresponding value for the random resistor network. Therefore, we conjecture that one goes from the scalar to the vector multifractal spectrum by simply translating the α axis by a constant amount (1, 0, or -1), depending on the ensemble chosen.

This conjecture *stricto sensus* applies to AE systems. However, it has been suggested by recent numerical simulations that the universality classes of CF and AE systems are identical.[12] Therefore, it should also apply to central-force percolation. In order to test both propositions, we studied numerically the CF case.

5. CHOICE OF THE "BEST" ENSEMBLE

The constant-current ensemble is certainly the best one in order to derive any critical exponent in the scalar case. This numerical observation comes from the fact that $x_c(n)$ is always finite for any n (i.e., the current in any bond is bounded by the current injected in the lattice and the fractal dimension of the bonds that carry this maximum current is nonzero). This is no longer the case for elasticity: Even if a unit force is applied onto the lattice, the maximum local force can be much larger due to lever-arm effects. Therefore, one is naturally led to use the possibility to define any ensemble which reproduces the constant-current ensemble property for elastic lattices, i.e., that the infinite-order scaling exponent is finite. In this spirit, let us introduce the a-ensemble, where the moments are computed according to

$$m_a(n) = \left\langle \left[\sum_k (\sqrt{e_k})^n \right] \Big/ E^{an} \right\rangle \propto L^{y_a(n)} \tag{11}$$

where E is the elastic energy of a lattice and $\langle \cdots \rangle$ denotes the average over lattices. Then we have

$$y_a(n) = x_c(n) + n[(a-1)\, x_c(2) + 2a - 1] \tag{12}$$

The "best" value of a, a^*, is such that $y_a(n)$ remains finite for n large; thus,

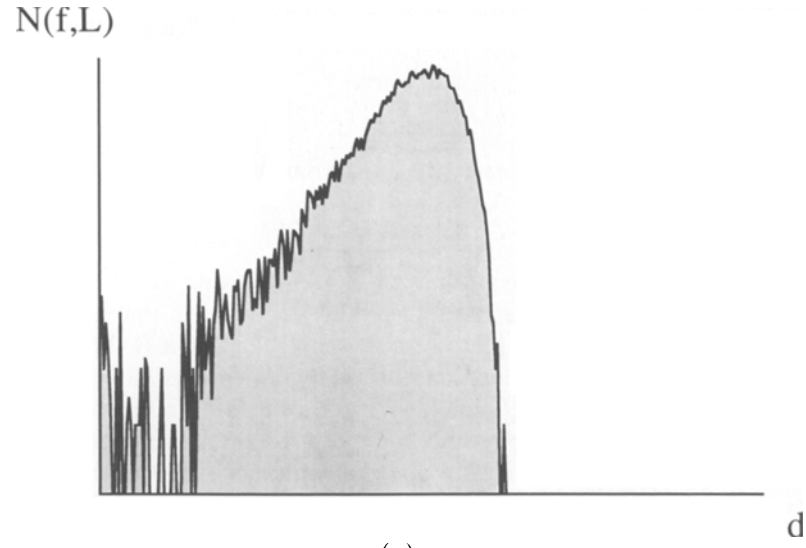
$$a^* = [x_c(2) + 1]/[x_c(2) + 2] = (t + \nu)/\tau \tag{13}$$

or numerically in two dimensions, $a^* \cong 2/3$. For this value of a^*, one recovers $y_{a^*}(n) = x_c(n)$. This ensemble should be the one that exhibits the least fluctuations for large-order moments and therefore it is the one to be chosen for numerical investigations of scaling exponents (for large-order moments). However, for the numerical result we present in the next section, we did not use this ensemble, since we did not expect these results at the time of the numerical computations. Nevertheless, we can expect that among the three distinct ensembles recorded, (force, energy, displacement) the second one should be the more precise one for large moments since the divergence of the exponents for large n is weaker than for other ensembles [see Eq. (9)]. We indeed observed such an effect.

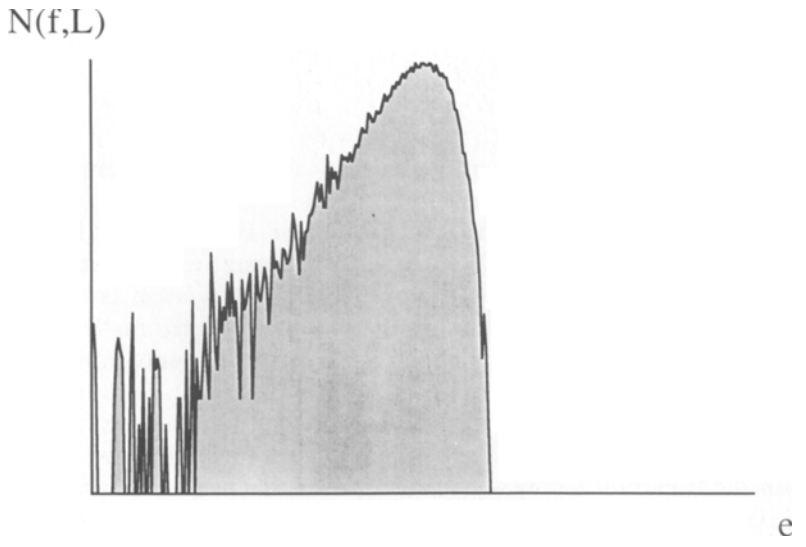
6. RESULTS

We now turn to the results of numerical simulations performed on central-force percolation lattices.

We study triangular lattices where a fraction $1 - p$ of bonds are cut at random. Each bond can be thought of as a spring freely rotating at its endpoints (i.e., nodes of the lattice). We consider $L \times L$ lattices with periodic boundary conditions in one direction (say, horizontally) and the two horizontal borders are attached to two rigid bars onto which a displacement is imposed. The fraction p of bonds present is chosen to be 0.642 ± 0.002, in accordance with previous results[12] on the determination of the critical density p^*. The distribution of forces on the lattice is obtained through a conjugate gradient method. The precision asked for was 10^{-10} for the usual stopping criterion (see, e.g., ref. 17). We checked the local equilibrium of each node and found that the maximum unbalanced force for the largest lattices considered (80×80) was less than 10^{-8}. We generated 500 30×30, 250 40×40, 200 50×50, 110 60×60, and 50 80×80 lattices. For each lattice size, we recorded both histograms and moments of the force distribution. Each of these were computed in the three ensembles mentioned above: constant displacement, energy, and force.

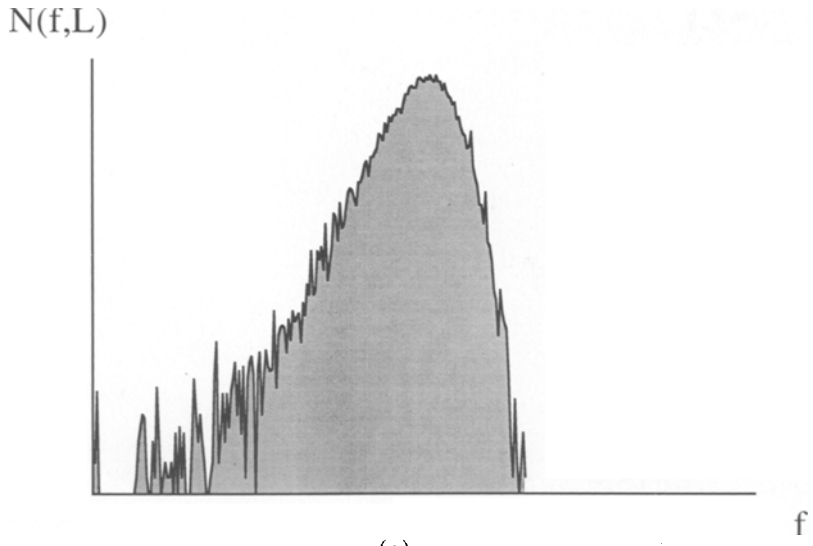


(a)



(b)

Fig. 1. The rescaled histograms of the force distribution in the central-force percolation problem at threshold in three different ensembles: constant force (f), energy (e), and displacement (d). These histograms should all tend toward the "multifractal spectrum" of the force distribution $f(\alpha)$. It is argued in the text that this spectrum can be obtained from that of the random resistor network at percolation threshold by a simple translation of the horizontal axis.



(c)

Fig. 1 (*continued*)

Figure 1 shows, for size 50, the three rescaled histograms, $\log[N(f, L)]/\log L$ plotted versus $\alpha(f, L) = \log f/\log L$. It is worth noting that the constant-force histogram is less well behaved for very large and very small forces. Due to lever-arm effects, the upper bound on the local microscopic force is equal to the force imposed on the lattice times the lattice size L. The fact that this upper limit increases with the lattice size is responsible for this poor behavior. This is equivalent to the divergence of the exponents $y_f(n)$ with increasing n. The two other ensembles share the property of having a fixed strict upper bound independent of lattice size. This imposes the constraint that the exponents $y_{ee}(n)$ and $y_d(n)$ tend to either a constant or minus infinity when n tends to plus infinity.

As mentioned in Section 2, an alternative way of obtaining the multifractal spectrum is to analyze the scaling of moments of forces. Figure 2 shows the evolution of the log of the first five moments as a function of lattice size L in a log–log plot. These are shown for $p = 0.641$, for it appears that this value gave better results (the data show less curvature) than did $p = 0.642$ for high-order moments. Due to the fact that most of these exponents are expected to be quite large, corrections to scaling appear to be suite important. In other words, the large moments (in particular) display a large cuvature in the log–log plot versus L. This is also known to be true for AE systems, where τ/ν, for example, the scaling exponent of the second moment, is still 0.5 below its estimated asymptotic value for

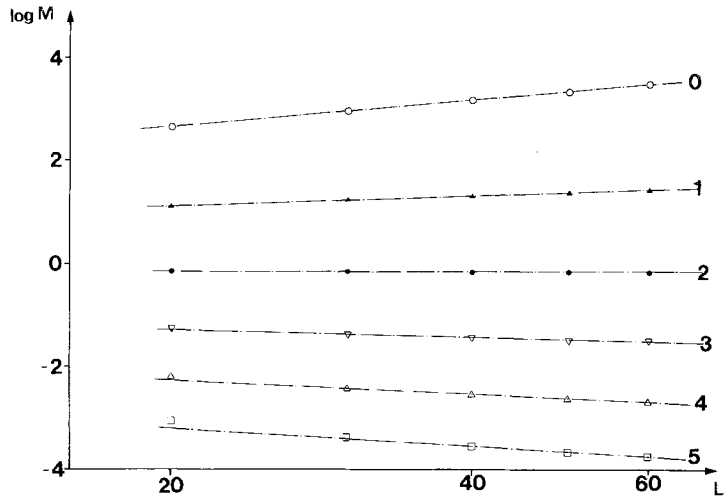


Fig. 2. Log–log plot of the moments of force distribution of the central-force percolation problem at threshold, versus the lattice size L, for the constant-energy ensemble.

$L = 100$.[6] The consequence of this fact is the large uncertainty in the exponents. This renders even more relevant the use of the best-adapted ensemble for a given moment.

We computed the different exponents $y(n)$ (reported in Table I) directly from the raw data, without using Eqqs. (5) and (6) relating them. We see, however, that these relations are fulfilled. Moreover, from the data concerning $x_c(n)$ (obtained numerically in a previous work[3]), we report also in Table I the value of the exponents $y(n)$ expected from our conjecture, Eq. (8). We see that the agreement is fairly good. We note that even the exponents relative to the larger-order moments are consistent with our prediction, although the error bars are large.

In a similar spirit, we have plotted in Fig. 3 the apparent exponents $y(n)$ in the three ensembles as a function of n. The data are always close to their expected value (dotted line). The asymptotic behavior for large n is also satisfactory. Here, too, the value of p is chosen to be 0.641, rather 0.642, since it gave better-quality fits.

It had already been reported[12] that the fractal dimension of the backbone $y(0)$, identical in the three ensembles in CF percolation was very close to that of scalar percolation, $x(0)$, as well as the elastic modulus, whose scaling seems identical for AE and CF systems. This can be interpreted as indicating that angular elasticity in CF was recovered at an intermediate length scale, larger than that of a bond. Indeed, from a close analysis of the distribution of forces in one backbone, it can be seen that

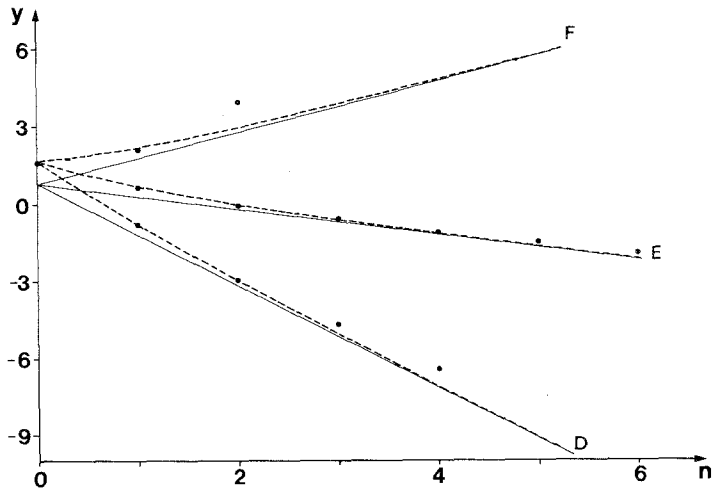


Fig. 3. Scaling exponents of the nth moment $y(n)$ versus n [Eq. (4)] for the three different ensembles considered in the text [constant force (f), energy (e), and displacement (d)] for the CF percolation problem. (●) The exponents obtained from the numerical simulations. (···). The predictions of Eqs. (8), (8′), and (8″). The asymptotic (large-n) theoretical behaviors are the straight lines. For $n=0$, $y(0)$ is the fractal dimension of the backbone, and $y(2)$ is τ/ν, 0, $-\tau/\nu$, respectively, in the three ensembles f, e, and d.

the torques are certainly responsible for the largest forces. However, due to the nonlocality of central force, it is not possible at the scale of bonds to picture the backbone by the nodes-links-blobs scheme (see, e.g., ref. 14). Therefore, the concept of singly connected bonds needs, at least, to be redefined. Our results indicate that, if we define them as the bonds that contribute to the large-order moments, they form a fractal set of dimensionality quite consistent with $1/\nu$. (Although we do not claim to estimate ν in this way, since it would be obtained through the difference of two critical exponents of large-order moments, for which the accuracy is very limited.)

7. CONCLUSION

We have presented arguments that the shape of the current distribution of the random resistor network at percolation threshold is identical to that of the force distribution in AE and CF elastic systems at their respective thresholds. The shapes of the f versus α curves of these problems are the same. The argument is supported by numerical studies of the CF percolation network and our data are good up to the fifth moment of the force distribution. Higher-order moments are very sensitive to statistical fluctuations, which, together with corrections to scaling, prevent any accurate determinations of critical exponents.

APPENDIX

If a unit current is sent through the lattice, then the resistance R of the lattice, as well as the voltage drop V, are equal to the second moment of the current distribution:

$$R = \left(\sum i^2 \right)_c \tag{A1}$$

where parentheses with subscript c (resp. e and v) refer to the unit current (resp. dissipation and voltage) boundary condition. If now we submit the same lattice to a unit voltage drop, then the nth moment is given by

$$\left(\sum i^n \right)_v = \left[\sum (i/R)^n \right]_c = \left(\sum i^n \right)_c \Big/ \left[\left(\sum i^2 \right)_c \right]^n \tag{A2}$$

Using the scaling relation (1), we can thus write

$$L^{x_v(n)} \propto L^{x_c(n)} \cdot L^{-x_c(2)} \tag{A3}$$

resulting in Eqs. (2). In a similar way, for the constant-dissipation ensemble we can write

$$\left(\sum i^n \right)_v = \left[\sum (i/\sqrt{R})^n \right]_c = \left(\sum i^n \right)_c \Big/ \left[\left(\sum i^2 \right)_c \right]^{n/2} \tag{A4}$$

which gives Eqs. (3).

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